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ON THE CONVERGENCE OF THE SERIES SOLUTION FOR A CYLINDRICAL SHELL SUBJECT TO A SEGMENTAL LINE LOAD

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BASED on the previous investigations [1, 2], and upon the physical application of certain loading system, a method is obtained for accelerating the convergence of the series solution for the problem of a circular cylindrical shell subject to a segmental longitudinal line load at its middle. This is achieved by eliminating certain sequences of terms in the series solution, thus increasing the gap between the successive terms. Assume that a simply supported circular cylindrical shell with length $2l$, and radius R is acted upon by an inward radial longitudinal line load of length $2d$, and intensity $\bar{p}$ = constant, at the middle of the shell. Choosing the origin at the middle of the line load, and employing Novozhilov's theory of cylindrical shells [3], one obtains the solution for the loaded region exactly as the one given in [1]. Thus, we have

$$\left.\begin{aligned}
\tilde{T} = {} & -\varepsilon(\theta) i b^2 \bar{p} \sin\theta + \frac{i b^2 \bar{p}}{2\pi}(\cos\theta + 2\theta\sin\theta) \\
& - \frac{i b^2 \bar{p} \xi^2}{2\pi(1 - i b^2)} \cos\theta + A_0 + C_0 \cosh\alpha_0\xi + (A_1 + C_1\cosh\alpha_1\xi)\cos\theta \\
& + \sum_{m=2}^{\infty} (A_m \cosh\beta_m\xi + C_m\cosh\alpha_m\xi)\cos m\theta, \\
& \pi \ge \theta \ge -\pi, \qquad \varepsilon(\theta) = \begin{cases} + & \text{for } \theta > 0 \\ - & \text{for } \theta < 0, \end{cases} \\
& \frac{d}{R} \ge \xi \ge -\frac{d}{R}.
\end{aligned}\right\} \tag{1}$$

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For the load-free region we obtain the following series solution which satisfies the simple support conditions [1] automatically:

$$\begin{aligned}\tilde{T}^* = {} & C_0^*\left(\tanh \alpha_0 \frac{l}{R}\cosh \alpha_0\xi - \sinh \alpha_0\xi\right) + A_0^*\left(\xi - \frac{l}{R}\right) \\ & + \left[C_1^*\left(\tanh \alpha_1 \frac{l}{R}\cosh \alpha_1\xi - \sinh \alpha_1\xi\right) + A_1^*\left(\xi - \frac{l}{R}\right)\right]\cos\theta \\ & + \sum_{m=2}^{\infty}\left[C_m^*\left(\tanh \alpha_m \frac{l}{R}\cosh \alpha_m\xi - \sinh \alpha_m\xi\right)\right. \\ & \left. + A_m^*\left(\tanh \beta_m \frac{l}{R}\cosh \beta_m\xi - \sinh \beta_m\xi\right)\right]\cos m\theta,\end{aligned} \tag{2}$$

in which the symbols with asterisks refer to the region $l/R \geq \xi \geq d/R$. It can be shown that, the continuity conditions at the circumferential lines $\xi = \pm d/R$ are equivalent to the following four complex relations.

$$\left.\begin{aligned}\tilde{T} = \tilde{T}^*, \quad \frac{\partial \tilde{T}}{\partial \xi} = \frac{\partial \tilde{T}^*}{\partial \xi}, \\ \tilde{T}_2 = \tilde{T}_2^*, \quad \frac{\partial \tilde{T}_2}{\partial \xi} = \frac{\partial \tilde{T}_2^*}{\partial \xi},\end{aligned}\right\} \text{ at } \xi = \pm\frac{d}{R}. \tag{3}$$

Applying expressions (1 and 2) in (3) one obtains the unknown constants of integration $A_0, C_0, \ldots A_m, C_m, A_0^*, C_0^*, \ldots A_m^*, C_m^*$. They shall not be given here for the sake of brevity.

Consider now the convergence of the series solution (1) for $\xi = \theta = 0$. It can be shown that A_m and C_m behave as $\bar{e}^{md/R}/m^2$ when m becomes large. Hence, for values of d/R not too small, the series solution (1) has fairly rapid convergence property. To accelerate the convergence of the series solution for small values of d/R, we apply an inward line load, acting at $\theta = \pi$, with length and intensity identical to the original load. Denoting the solution for this loading by $\tilde{T}_\pi$, and superposing it to $\tilde{T}$, we obtain a series which has only terms for even m:

$$(\tilde{T} + \tilde{T}_\pi)_{\theta=0} = 2\left[A_0 + C_0\cosh\alpha_0\xi + \sum_{m=2,4,6,\ldots}^{\infty}(A_m\cosh\beta_m\xi + C_m\cosh\alpha_m\xi)\right]. \tag{4}$$

In order to remove the effect of the segmental load acting at $\theta = \pi$, we apply a load in the opposite direction, employing the solution obtained in investigation [2]. Thus, we have

$$\left.\begin{aligned}-(\tilde{T}_\pi)_{\theta=0} = \sum_{n=1,3,5,\ldots}^{\infty} & \left[\frac{-ib^2Rp_n^*(\bar{\alpha}_n^2 - \bar{m}^2)}{\bar{\alpha}_n\bar{D}_n\sinh\bar{\alpha}_n\pi}\cosh 2\pi\bar{\alpha}_n\right. \\ & \left. + \frac{ib^2Rp_n^*(\bar{\beta}_n^2 - \bar{m}^2)}{\bar{\beta}_n\bar{D}_n\sinh\bar{\beta}_n\pi}\cosh 2\pi\bar{\beta}_n\right]\sin\frac{\bar{m}}{2}\left(\frac{l}{R} + \xi\right),\end{aligned}\right\} \tag{5}$$

in which the quantity p_n^* [2] should be considered for an inward line load. Combination of (4 and 5) now gives

$$\left.\begin{aligned}(\tilde{T})_{\theta=0} &= 2(A_0+C_0\cosh\alpha_0\xi)+2\sum_{m=2,4,6,\ldots}^{\infty}(A_m\cosh\beta_m\xi+C_m\cosh\alpha_m\xi)\\ &\quad-\sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{-ib^2Rp_n^*(\bar{\alpha}_n^2-\bar{m}^2)}{\bar{\alpha}_n\bar{D}_n\sinh\bar{\alpha}_n\pi}\cosh 2\pi\bar{\alpha}_n\right.\\ &\quad\left.+\frac{ib^2Rp_n^*(\bar{\beta}_n^2-\bar{m}^2)}{\bar{\beta}_n\bar{D}_n\sinh\bar{\beta}_n\pi}\cosh 2\pi\bar{\beta}_n\right)\sin\frac{\bar{m}}{2}\left(\frac{l}{R}+\xi\right).\end{aligned}\right\}\tag{6}$$

The first series in the right-hand side of (6) has a better convergence property than the original series (1), while the second one converges very rapidly [2]. Similarly, one removes terms corresponding to $m = 2, 6, 10, \ldots$, etc., in relation (6), by adding and eliminating segmental loads of intensities $2\bar{p}$ at $\theta = \pi/2$. Thus, we obtain

$$\left.\begin{aligned}(\tilde{T})_{\theta=0} &= -ib^2\bar{p}+4(A_0+C_0\cosh\alpha_0\xi)+4\sum_{m=4,8,12,\ldots}^{\infty}(A_m\cosh\beta_m\xi+C_m\cosh\alpha_m\xi)\\ &\quad-\sum_{n=1,3,5,\ldots}^{\infty}\left\{-\frac{ib^2Rp_n^*(\bar{\alpha}_n^2-\bar{m}^2)}{\bar{\alpha}_n\bar{D}_n\sinh\bar{\alpha}_n\pi}\left[\cosh 2\pi\bar{\alpha}_n+2\cosh\frac{3\pi}{2}\bar{\alpha}_n\right]\right.\\ &\quad\left.+\frac{ib^2Rp_n^*(\bar{\beta}_n^2-\bar{m}^2)}{\bar{\beta}_n\bar{D}_n\sinh\bar{\beta}_n\pi}\left[\cosh 2\pi\bar{\beta}_n+2\cosh\frac{3\pi}{2}\bar{\beta}_n\right]\right\}\sin\frac{\bar{m}}{2}\left(\frac{l}{R}+\xi\right).\end{aligned}\right\}\tag{7}$$

The procedure can be continued in the same fashion. For example, terms corresponding to $m = 4, 12, 20, \ldots$, etc., in (7) are eliminated by applying and removing segmental loads with intensities $2\bar{p}$ at $\theta = \pi/4$ and $\theta = 3\pi/4$. The above technique of improvement of convergence could also be employed for any point with circumferential coordinate $\theta \neq 0$. Thus, one applies and eliminates segmental loads at positions $\pi-\theta$, $\pi/2+\theta$, $\pi/2-\theta, \ldots$, etc.

It is interesting to determine under what circumstances the aforementioned technique gives its best efficiency. The formulas show that for the values $d/R-\xi > \pi$, one should use formula (1) without the improvement of convergence. This follows from the fact that the ratio of the absolute values of two successive terms in the second series of (6) is $\bar{e}^{\pi}$ for large n. For values $\pi \geq d/R-\xi \geq \pi/2$, one employs formula (6), while for $\pi/2 \geq d/R-\xi \geq \pi/8$, he utilizes (7). In order to show the validity of solutions (1 and 2), the values of T_ξ and T_θ, calculated on the basis of these formulas, are compared with those obtained in investigation [2]. The results are presented in Table 1. In Tables 2 and 3 the individual terms for $\tilde{T}$, obtained from (6 and 7) are given for different values of d/R to demonstrate the rapid convergence of the series solution.

TABLE 1. COMPARISON OF NONDIMENSIONAL STRESS RESULTANTS BASED ON THE SOLUTION IN THE CURRENT INVESTIGATION AND THAT OF [2] FOR $l/R = \pi$, $b = 5$ and $d/R = \pi/5$

	Method of current investigation	Method of investigation [2]
$T_\xi/\bar{p}$	$-2{\cdot}559576$	$-2{\cdot}559996$
$T_\theta/\bar{p}$	$-0{\cdot}795843$	$-0{\cdot}788422$

TABLE 2. REAL PARTS OF TERMS IN SERIES (6) FOR $l/R = 2{\cdot}5$, $b = 5$ AND $d/R = 0{\cdot}75$

m	First series	n	Second series
2	−3·1268	1	0·50356
4	−0·41514	3	−0·017239
6	−0·046488	5	−0·00058227
8	−0·0055726	7	0·000004366
10	−0·00075396	9	0·000001238
12	−0·000111770	11	0·000000053

TABLE 3. REAL PARTS OF TERMS IN SERIES (7) FOR $l/R = 2{\cdot}5$, $b = 5$ AND $d/R = 0{\cdot}375$

m	First series	n	Second series
4	−0·98584	1	−1·78440
8	−0·058528	3	−0·15904
12	−0·0053736	5	−0·000031499
16	−0·00064204	7	−0·00016019
20	−0·000088804	9	−0·000017952
24	−0·000013459	11	0·000035008

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